MATHEMATICS TEACHER
Learning & Teaching PK-12

Mathematics Teacher: Learning and Teaching PK-12, is NCTM's newest journal that reflects the current practices of mathematics education, as well as maintains a knowledge base of practice and policy in looking at the future of the field. Content is aimed at preschool to 12th grade with peer-reviewed and invited articles. *MTLT* is published monthly.

ARTICLE TITLE:

Robot Kinematics Tasks: Geometry in Motion

AUTHOR NAMES:

Williams, Jedediyah; and Pashukov, Vladimir

DIGITAL OBJECT IDENTIFIER: 10.5951/MTLT.2025.0385

VOLUME: 119

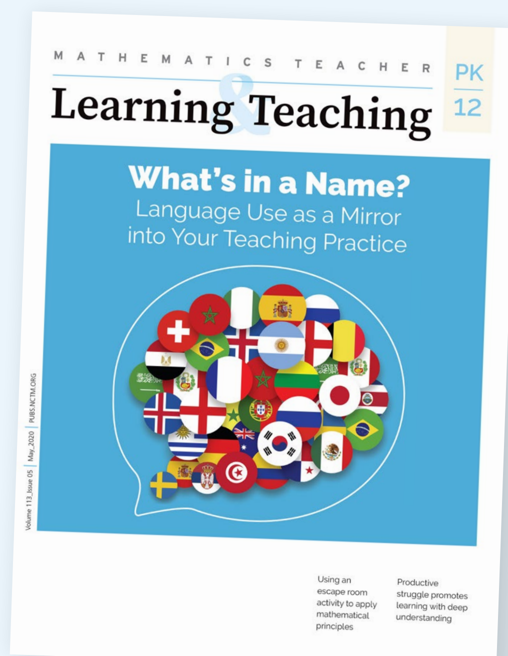
ISSUE NUMBER: 7

Mission Statement

The National Council of Teachers of Mathematics advocates for high-quality mathematics teaching and learning for each and every student.

Approved by the NCTM Board of Directors on July 15, 2017.

CONTACT: mtlt@nctm.org



NCTM NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Access digital content at nctm.org/mtlt11907at95.

Robot Kinematics Tasks: Geometry in Motion

Robot kinematics provide a simple, authentic context with beautiful, rich mathematics. These tasks bridge algebra and geometry, with extensions into advanced topics.

Jedediyah Williams and Vladimir Pashukov

A primary application of trigonometry is kinematics, the geometry of modeling motion without forces. Kinematics is fundamental to physics, engineering, robotics (Craig, 2022; Williams, 2024), computer graphics (Russell, 2009), and any field that models motion. In this article, we introduce four authentic robot kinematics tasks to support trigonometry and related geometry topics.

We model a robotic arm as a series of links connected by joints (Figure 1). Each link i has length ℓ_i and a joint angle θ_i measured relative to the preceding link, or relative to the horizontal in the case of link 1. The end-effector is the tool at the end of the last link, pictured in Figure 1 as a "gripper."

The set of joint angles are parameters that correspond to configurations of the robot. For a given set of joint angles, forward kinematics is the problem of determining the location of the gripper (the coordinate of the point on the gripper's palm throughout Figure 1). The reverse process, determining a set of joint angles to achieve a desired position of the gripper, is inverse kinematics.

The workspace of a robot is the set of spatial coordinates containing every point reachable by the gripper. For example, the workspace of a 1-link robot is the circle of radius ℓ_1 that it can inscribe. Finally, kinematic control involves determining functions that will place a robot in particular configurations over time.

The simple context of a robotic arm connects a broad set of content standards (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA Center & CCSSO], 2010) and emphasizes Standards for Mathematical Practice 2—*Reason abstractly and quantitatively*—and 4—*Model with mathematics*. These tasks have appeared within a precalculus curriculum to facilitate units on polar coordinates, vectors, matrix operations, functions, and derivatives.

TASK 1: FORWARD KINEMATICS

The forward kinematics problem, finding the location of the gripper given a set of angles and link lengths, is introduced by analyzing a 1-link robotic arm. This connects directly to the unit circle scaled by link length ℓ_1 (HSF.TF.A; NGA Center & CCSSO, 2010).

After familiarizing themselves with the problem, and possibly reviewing how to turn each problem into a right triangle to be solved (HSG.SRT.C and HSF.TF.A; NGA Center & CCSSO, 2010), students work to complete the table in Figure 2 with the (x, y) coordinate p_1 of each gripper.

The 1-link forward kinematics problem is considered solved when we have reflected on patterns and generalized the location p_1 at the end of the first link:

$$p_1 = (\ell_1 \cos(\theta_1), \ell_1 \sin(\theta_1))$$

Precalculus students familiar with vectors or polar coordinates may recognize this formulation.

Visualizations of a 1-link robot in Desmos are animated using sliders for ℓ_1 and θ_1. This provides practice animating in Desmos while priming the first step for later modeling a 2-link. The addition of a gripper into their animation is posed as a challenge for students.

We approach robots with more links by repeating the 1-link process from the first to the last link. Consider the robot with $\ell_1 = 2$ and $\ell_2 = 1$. A forward kinematics problem might ask for the robot's gripper location given angles $\theta_1 = \frac{\pi}{3}$ and $\theta_2 = \frac{-\pi}{4}$. With the first link starting from the origin, the end point of link 1, p_1, is located at the coordinate:

$$p_1 = \left(2\cos\left(\frac{\pi}{3}\right), 2\sin\left(\frac{\pi}{3}\right)\right) = \left(1, \sqrt{3}\right)$$

It is from this point that we initiate link 2. The end point of link 2, p_2, is where we would find the gripper for a 2-link, and it is located at the coordinate:

$$p_2 = \left(2\cos\left(\frac{\pi}{3}\right) + 1\cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right), 2\sin\left(\frac{\pi}{3}\right) + 1\sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)\right) \approx (1.966, 1.991)$$

Figure 1 Examples of 1-, 2-, and 3-Link Robotic Arms in 2D

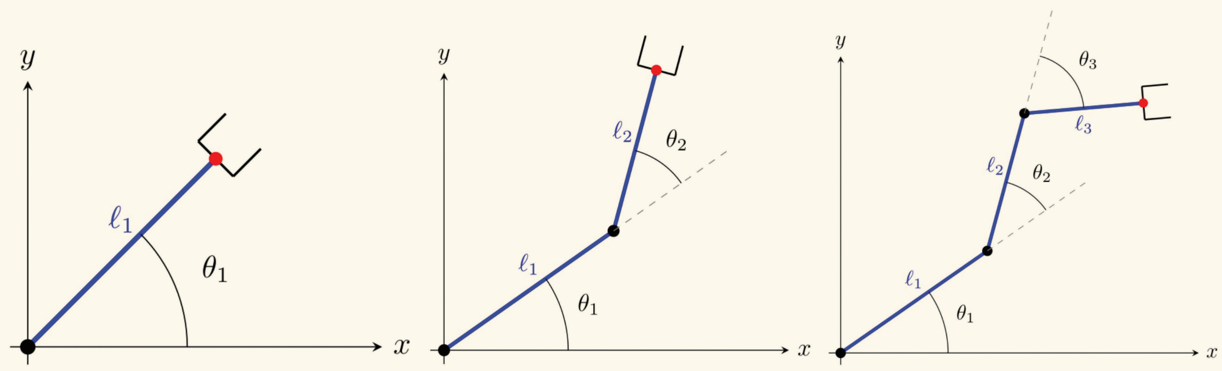


Figure 2 A Student's Table Combines Exact Values and Approximations for p_1 Gripper Coordinates Given Lengths and Angles

	$\ell_1 = 0.7$	$\ell_1 = 1$	$\ell_1 = 2$	$\ell_1 = 10$
$\theta_1 = -\pi/4$	(0.495, −0.49)	$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$	$(\sqrt{2}, -\sqrt{2})$	$(5\sqrt{2}, -5\sqrt{2})$
$\theta_1 = 0$	(0.7, 0)	(1, 0)	(2, 0)	(10, 0)
$\theta_1 = \pi/4$	(0.495, 0.495)	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	$(\sqrt{2}, \sqrt{2})$	$(5\sqrt{2}, 5\sqrt{2})$
$\theta_1 = \pi/3$	(0.35, 0.61)	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	$(1, \sqrt{3})$	$(5, 5\sqrt{3})$
$\theta_1 = \pi$	(−0.7, 0)	(−1, 0)	(−2, 0)	(−10, 0)

Jedediyah Williams, jedediyah@gmail.com, teaches high school in Belmont, MA. He is interested in computing and experimental mathematics in education.

Vladimir Pashukov is a senior at Belmont High School, interested in robotics and applied physics. He plans to attend Washington University for computer engineering.

doi:10.5951/MTLT.2025.0385

where $\frac{\pi}{3}-\frac{\pi}{4}$ is the total rotation (i.e., the sum of the angles $\theta_1+\theta_2$).

Students work to solve instances of the forward kinematics problem on paper. We then generalize the 2-link problem and determine p_2 in terms of lengths and angles:

$$p_2 = (\ell_1 \cos(\theta_1) + \ell_2 \cos(\theta_1+\theta_2), \ell_1 \sin(\theta_1) + \ell_2 \sin(\theta_1+\theta_2))$$

and model the 2-link with sliders in Desmos (Figure 3).

Once students have solved a 2-link arm, we explore ways to generalize to n links, which motivates discussion of notation. For students familiar with matrices, we translate our formulas from above into homogeneous transformation matrices, which combine nonlinear rotations and linear translations within matrix multiplication. We represent the rotation of angle θ_i and the translation of link length ℓ_i as follows:

$$\mathbf{R}_i = \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i) & 0 \\ \sin(\theta_i) & \cos(\theta_i) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{T}_i = \begin{bmatrix} 1 & 0 & \ell_i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so that multiplying $\mathbf{R}_1\mathbf{T}_1$, for example, results in the transformation matrix:

$$\mathbf{R}_1\mathbf{T}_1 = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \ell_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & \ell_1\cos(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) & \ell_1\sin(\theta_1) \\ 0 & 0 & 1 \end{bmatrix}$$

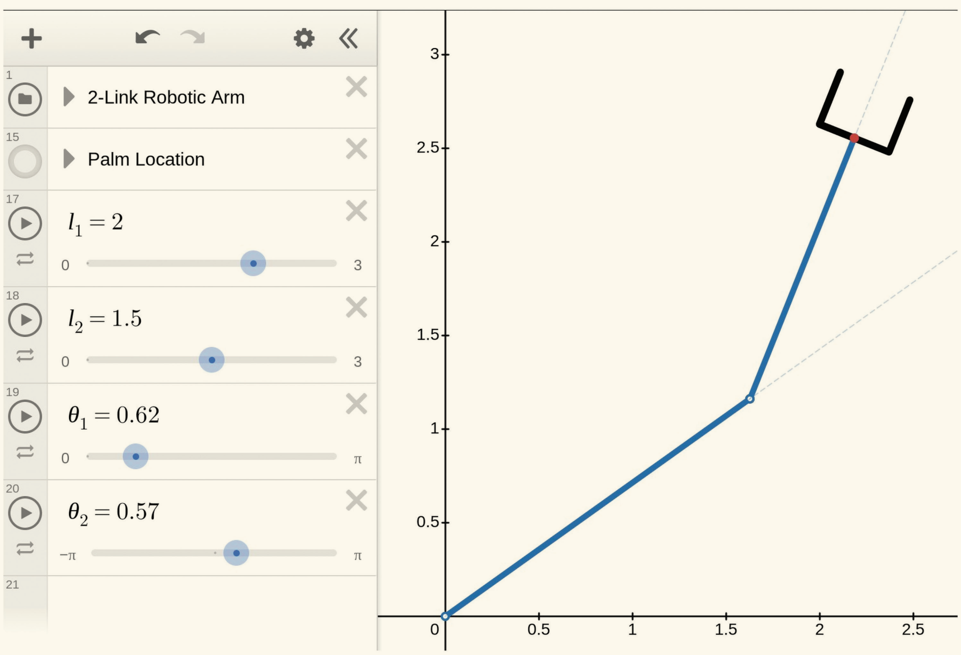
The first two terms in the last column correspond to the coordinate at the end of the 1-link transformation. The coordinates of p_2 for a 2-link robot can be found in the product:

$$\mathbf{R}_1\mathbf{T}_1\mathbf{R}_2\mathbf{T}_2 = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \ell_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) & 0 \\ \sin(\theta_2) & \cos(\theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \ell_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & \ell_1\cos(\theta_1)+\ell_2\cos(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & \ell_1\sin(\theta_1)+\ell_2\sin(\theta_1+\theta_2) \\ 0 & 0 & 1 \end{bmatrix}$$

The last column of this transformation matrix contains the x and y components of the formula for p_2 from above. This pattern of multiplying by subsequent rotations and translations continues for arbitrarily

Figure 3 Simulation of a 2-Link Robotic Arm



many links, so that the transformation from the origin to the end of the nth link of a robot is found within the product:

$$\prod_{i=1}^{n} \mathbf{R}_i \mathbf{T}_i$$

The resulting matrix is composed of the rotation matrix and translation vector from the origin to the end of the nth link. There is vastly more to be said about this transformation, but we leave it here for now.

TASK 2: WORKSPACE GEOMETRY

The workspace of a robot is the set of coordinates reachable by its gripper. We prompt and offer feedback as we construct workspace understanding through guiding questions (see the supplementary material [link online]):

1. What is the workspace of a 1-link arm?
2. What is the workspace of a 1-link arm with the constraint $\frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6}$?
3. What is the workspace of a 2-link arm with lengths 1 and 1?
4. What is the workspace of a 2-link arm with lengths 1 and 2?
5. What is the workspace of a 2-link arm with lengths 2 and 1?
6. Define an arm with the workspace $x^2 + y^2 \leq 3^2$.
7. A 1-link arm with length 1.5 m has the ability to translate horizontally as much as 2 m to the right. Find its workspace.

Determining workspace is an important problem in robotics. Here, it serves primarily as a conceptual primer for our next task.

TASK 3: INVERSE KINEMATICS

The forward kinematics problem asked for the position of the gripper given a set of angles. The inverse kinematics problem asks for a set of angles that place the gripper at a target position. This problem may or may not have a solution, depending on whether the target position is within the workspace. If a position is reachable, then there may be multiple and even infinitely many solutions.

We initiate this task again with a 1-link arm, prompting students to determine the angle for the gripper to reach target positions (Figure 4). This exercise is equivalent to finding the standard position angle in a scaled unit circle, which could be broken down into triangles.

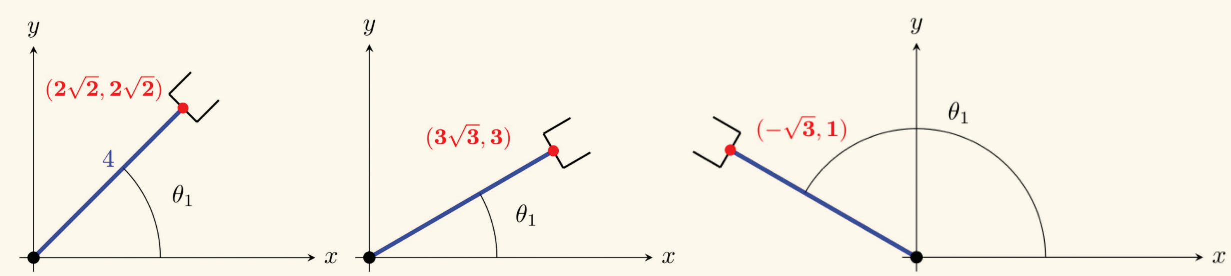
Once we feel comfortable finding the angle for a single link, we proceed to the 2-link. A common approach to solving robotics problems is to define the constraints and look for conditions that satisfy those constraints. For the 2-link inverse kinematics problem, our constraints are as follows:

1. Link 1 starts at the origin.
2. Link 2 ends at the target position.
3. Links 1 and 2 connect in between.

A geometric insight into satisfying these constraints is that we can search for solutions where the circle of radius ℓ_1 about the origin intersects the circle of radius ℓ_2 about the target position (Figure 5). Generally, these circles could intersect at 0, 1, or 2 points.

In the example in Figure 5, a circle of radius $\ell_1 = 2$ was constructed about the origin and a circle of radius $\ell_2 = 1$ was constructed about a target point $p_2 = (1.3, 2.1)$. The points of intersection of the circles correspond to two possible locations for p_1 that configure the gripper to reach p_2. The values of θ_1 and θ_2 labeled in Figure 5

Figure 4 Example Target Positions for a 1-Link Arm, With Solutions of $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{5\pi}{6}$



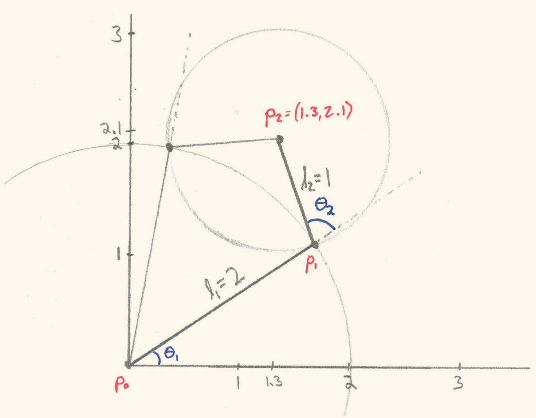
correspond to one of the two solutions of this inverse kinematics problem.

Finding intersection points along with the angles θ_1 and θ_2 that orient the links to meet at one of those points involves a substantial amount of algebra. This could be a task in itself, or it could be scaffolded with a graphing calculator to find approximate coordinate values.

When we move to the 3-link inverse kinematics problem, we discover that there can be an infinite number of solutions. Figure 6 depicts a solution approach for a 3-link arm with lengths $\ell_1 = 3$, $\ell_2 = 2$, and $\ell_3 = 1$ to reach a target location (4,2). We construct a circle of radius ℓ_1 around the origin to represent all of the possible positions p_1, and a circle of radius ℓ_3 around our target location to represent all the possible positions p_2. The next step toward a solution is to connect these two circles by a length of ℓ_2. While not every pair of points on the two circles is ℓ_2 apart, there are infinitely many pairs of points p_1 and p_2 on these circles that are exactly ℓ_2 apart! This can be observed by connecting a length 2 segment between the circles and then sliding that segment along their perimeters. There are more degrees of freedom in this case than there are constraints.

One approach to simplifying inverse kinematics for many links is to "lock" some of the joints in place to reduce the number of solutions. For example, we could choose to always constrain θ_1 to multiples of $\frac{\pi}{4}$. This is not necessary, nor is it always a reasonable choice, but it can reduce the number of solutions from infinite to finite for a 3-link arm.

Figure 5 A Sketch Exploring an Inverse Kinematics Problem With Two Solutions



Closed form solutions can be determined for a limited number of robotic arms. In practice, numerical methods are used with heuristics that optimize some aspect of the problem (i.e., computational efficiency or "nice" mathematical properties).

TASK 4: KINEMATIC CONTROL

The problem of (position-based) kinematic control involves building functions (HSF.BF.A.1; NGA Center & CCSSO, 2010) that relate joint angles over time. We refer to these functions as joint angle trajectories, and they can be written as $\theta(t)$ in cases when we want to start the robot at an initial angle $\theta(t_i)$ and move it to a final angle $\theta(t_f)$ over the time interval $[t_i, t_f]$.

Consider a 1-link arm that we want to move from 1 radian to 3 radians, starting at time $t_i = 2$ and completing the move at time $t_f = 8$. A first approach is to find the solution of the linear model (HSA.REI.C; NGA Center & CCSSO, 2010):

$$\theta(t) = a_0 + a_1 t$$
$$\theta(2) = 1$$
$$\theta(8) = 3$$

The first equation is our model, and the next two equations are constraints on our model. The solution $\theta(t) = \frac{1}{3} + \frac{1}{3}t$ depicted in Figure 7 is a linear interpolation, a curve-fitting construction that joins the initial and final points with a continuous linear function.

The linear solution runs into an important practical problem that has a mathematical solution: For a real robotic link, instantaneous starts will wear out the motor. This can be avoided if we "ramp up" the motion with a function that smoothly increases the slope $\theta'(t_i)$ from zero. One candidate model is a quadratic function with a vertex at the initial position. A quadratic interpolation with ramp up (Figure 8) is found by solving the following equation:

$$\theta(t) = a_0 + a_1 t + a_2 t^2$$
$$\theta'(2) = a_1 + 2a_2(2) = 0$$
$$\theta(2) = 1$$
$$\theta(8) = 3$$

Figure 6 Constructing Inverse Kinematics Solutions for a 3-Link Robotic Arm to Reach $p_3 = (4,2)$

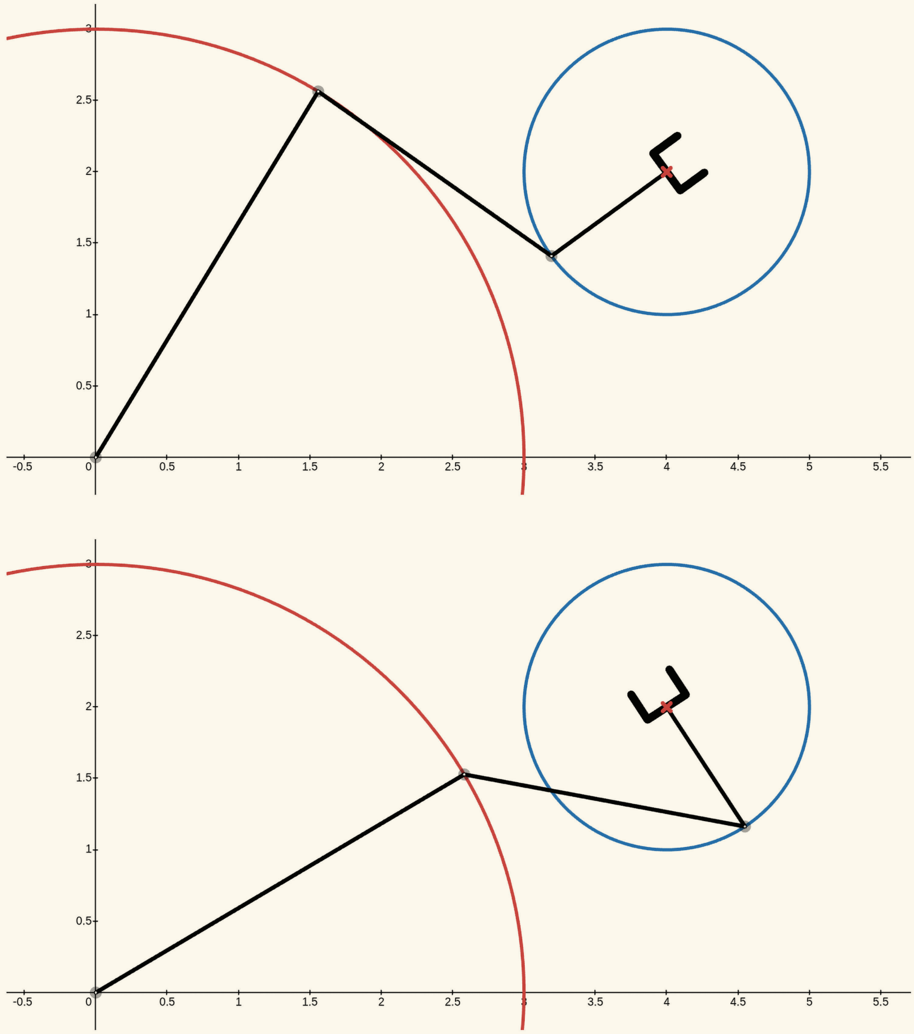
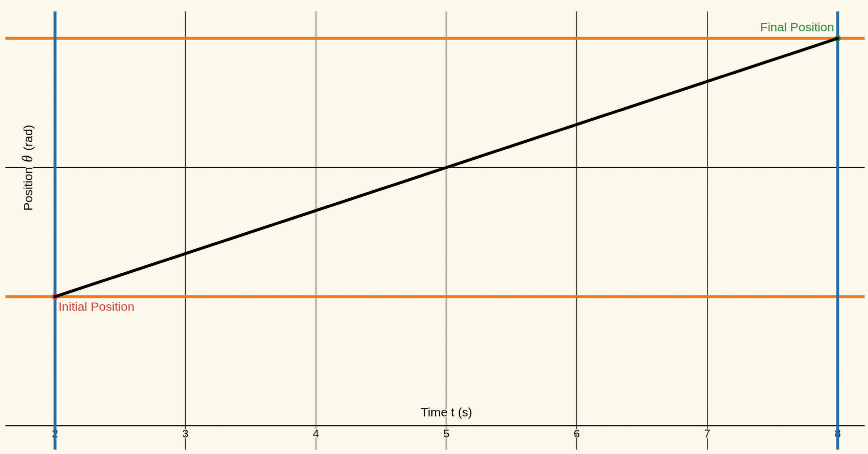


Figure 7 Linear Joint Angle Trajectory, $\theta(t) = \frac{1}{3} + \frac{1}{3}t$



where the second equation represents the constraint that the derivative of $\theta(t)$ is zero at time $t_i = 2$.

Naturally, we may want to also ramp down, which requires the additional constraint that $\theta'(t_f)$ approaches zero from the left. The trajectory in Figure 9 depicts the solution of cubic interpolation:

$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$
$$\theta'(2) = a_1 + 2a_2(2) + 3a_3(2)^2 = 0$$
$$\theta'(8) = a_1 + 2a_2(8) + 3a_3(8)^2 = 0$$
$$\theta(2) = 1$$
$$\theta(8) = 3$$

Figure 8 Quadratic Joint Angle Trajectory With Ramp Up, $\theta(t) = \left(\frac{11}{9}\right) + \left(\frac{-2}{9}\right)t + \left(\frac{1}{18}\right)t^2$

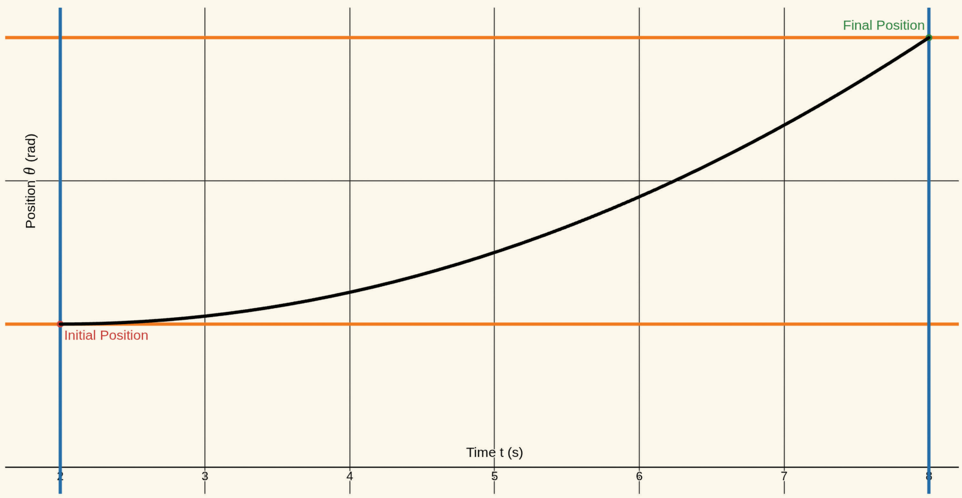
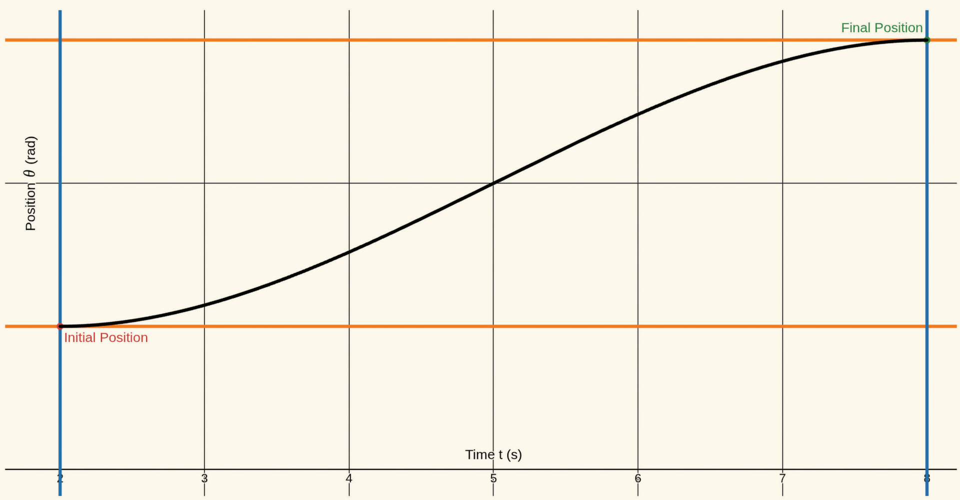


Figure 9 Cubic Joint Angle Trajectory With Ramp Up and Ramp Down, $\theta(t) = \left(\frac{49}{27}\right) + \left(\frac{-8}{9}\right)t + \left(\frac{5}{18}\right)t^2 + \left(\frac{-1}{54}\right)t^3$



Simulating a 1-link arm that follows various joint angle trajectories is insightful and can build intuition regarding the relationships of graphs to physical contexts (Hale, 2000). We experiment with non-polynomial trajectories (HSF.LE.A and HSF.LE.B; NGA Center & CCSSO, 2010) and consider how controlling two joints simultaneously can be used to trace patterns in 2D. For example, what joint trajectories for θ_1 and θ_2 would move the gripper in a circle around a given x, y coordinate?

CONCLUSION

The authentic application of a robotic arm contextualizes content from algebra through geometry into calculus and linear algebra, with very little overhead of additional information. These tasks are easily scaffolded and conceptually accessible at an introductory level and lead quickly into interesting, advanced topics to be explored.

REFERENCES

Craig, J. J. (2022). *Introduction to robotics: Mechanics and control* (4th ed.). Pearson.

Hale, P. (2000). Connecting research to teaching: Kinematics and graphs: Students' difficulties and CBLs. *Mathematics Teacher, 93*(5), 414-417. https://doi.org/10.5951/MT.93.5.0414

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. https://corestandards.org/Math

Russell, C. (2009). *Computer animation*. National Council of Teachers of Mathematics. https://www.nctm.org/Classroom-Resources/Illuminations/Lessons/Computer-Animation/

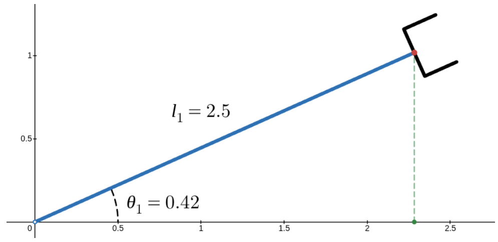
Williams, J. (2024). *The mathematics of robotics* [Conference presentation]. National Council of Teachers of Mathematics Virtual Conference. https://jedediyah.github.io/nctm2024virtual/

Robot Kinematics Tasks

Days 1 and 2. Forward Kinematics

A 1-link robotic arm contextualizes right triangle trigonometry problems. Finding the length of any side of an abstract right triangle can be replaced by asking for:

- The length of the robotic arm (hypotenuse)
- The x-coordinate of the gripper (adjacent)
- The y-coordinate of the gripper (opposite)

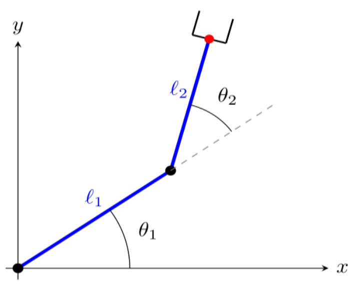


1. Review right triangle trigonometry and provide a worked example solving a 1-link arm **forward kinematics** problem: determining location of the gripper given length and angle.
2. Students work together to sketch the 1-link robot and find p_1, the location of the gripper, for pairs of given lengths $\ell_1 = \{0.7, 1, 2, 10\}$ and angles $\theta_1 = \{ \frac{-\pi}{4}, 0, \frac{\pi}{4}, \frac{-\pi}{3}, \pi \}$.
3. As a class, students pick random lengths and angles to input into a 1-link Desmos model. From these student-chosen values, they work to find p_1.
4. Once students feel confident with finding p_1, we outline the general solution $p_1 = (\ell_1 \cos(\theta_1), \ell_1 \sin(\theta_1))$ and demonstrate that this works for angles beyond 90^o.
5. Students break into groups at whiteboards and work through finding the gripper location for a 3-link (without explicit instruction on how to achieve this). If needed, we outline a solution strategy: go point by point and turn each link into its own triangle to be solved.

Modeling a Robot in Desmos

Students work to model either the 1-link or 2-link arm in Desmos. We review the formula to find p_1 for the 1-link and derive the general formula for the gripper location of a 2-link robot:

$$p_2 = (\ell_1 \cos(\theta_1) + \ell_2 \cos(\theta_1 + \theta_2) , \ell_1 \sin(\theta_1) + \ell_2 \sin(\theta_1 + \theta_2))$$



We challenge students to model a 2-link with a gripper in Desmos themselves. This task is scaffolded with the document "Guide: Modeling a Robotic Arm in Desmos," which walks through building a 2-link robot. Stylizing and labeling is an important component of the task, along with organizing the item pane in Desmos.

Resources

- 1-Link Robot: https://www.desmos.com/calculator/lz3h4evbxe
- 2-Link Robot: https://www.desmos.com/calculator/kpkbvwhll7